

Z_2 index for gapless fermionic modes in the vortex core of three dimensional paired Dirac fermions

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We consider the gapless modes along the vortex line of the fully gapped, momentum independent paired states of three dimensional Dirac fermions. For this we require the solution of fermion zero modes of the corresponding two dimensional problem in the presence of a point vortex, in the plane perpendicular to the vortex line. Based on the spectral symmetry requirement for the existence of the zero mode, we identify the generalized Jackiw-Rossi Hamiltonians for different paired states. A four dimensional generalized Jackiw-Rossi Hamiltonian possesses spectral symmetry with respect to an anti-unitary operator, and gives rise to a single zero mode only for the odd vorticity, which is described by a Z_2 index. In the presence of generic perturbations such as chemical potential, Dirac mass and Zeeman coupling, the two dimensional problem for the odd parity topological superconducting state maps onto two copies of generalized Jackiw-Rossi Hamiltonian, and consequently an odd vortex binds two Majorana fermions. In contrast there are no zero energy states for the topologically trivial s-wave superconductor, in the presence of any chiral symmetry breaking perturbation in the particle-hole channel, such as Dirac mass. We show that the number of one dimensional dispersive modes along the vortex line is also determined by the index of the associated two dimensional problem. For an axial superfluid state in the presence of various perturbations, we discuss the consequences of the Z_2 index on the anomaly equations.

I. INTRODUCTION

The existence of zero modes for the Dirac fermions in the presence of a topologically non-trivial configuration of an order parameter or a gauge field is an interesting problem in condensed matter and high energy physics. In a seminal paper, Jackiw and Rebbi demonstrated the emergence of the fermion zero modes on various defects in odd spatial dimensions¹. In particular, they have showed the existence of zero mode for a domain wall in one dimension, t'Hooft-Polyakov monopole and dyon in three dimensions. The fermion zero modes due to a solitonic defect can give rise to induced quantum numbers and fractionalization^{1,2}. The domain wall of a scalar order parameter (Dirac mass) in one dimension binds single zero mode, leading to the *fractionalization of charge*, which has been observed experimentally in polyacetylene³. Interestingly, the edge and the surface states of many gapped topological systems in higher dimensions are also determined by one or multiple copies of the one dimensional Jackiw-Rebbi model. Some specific examples are the edge states of the $d + id$ ⁴ and the $p + ip$ ^{5,6} superconductors and the quantum spin Hall state⁷ in two dimensions, and the surface states of the three dimensional topological insulators and superconductors⁸. For these higher dimensional problems, the domain wall describes a boundary between topologically distinct vacua. It is also interesting to note that the chiral surface states of gapless Weyl semi-metal and superconductors are also governed by similar one dimensional problem⁹.

The point vortex and the line vortex of a $U(1)$ symmetry breaking order parameter respectively in two and three dimensions, are also interesting topological defects.

The point vortices of the $p + ip$ superconductor^{5,10,11} and the Kekule bond-density wave order in graphene¹² support fermion zero modes. In contrast the line vortex of many three dimensional paired states such as axial superfluid¹³, axial superconductor¹⁴, superfluid $^3\text{He} - B$ ¹⁵, the gapped superconducting states of three dimensional Dirac fermions^{16–19} support one dimensional gapless fermions along the vortex core. If we consider, the line vortex along the \hat{z} direction, the momentum k_z is a conserved quantity, and the dispersing states are localized in the $x - y$ plane. Therefore, for $k_z = 0$, we indeed solve a two dimensional problem of the fermion zero modes. Consequently, the number of dispersing modes becomes equal to the number of fermion zero modes of an effective two dimensional problem in the presence of a point vortex.

The problem of zero energy modes bound by a point vortex in two dimensions has been considered by Jackiw and Rossi^{20,21}. The pertinent Hamiltonian is

$$H_{JR}[\mathbf{a}] = \sum_{j=1,2} \Gamma_j (-i\partial_j + i\Gamma_3 \Gamma_4 a_j) + \Delta_1 \Gamma_3 + \Delta_2 \Gamma_4, \quad (1)$$

where ∂_j 's are the derivatives with respect to the spatial coordinates, and Δ_1, Δ_2 are the components of a complex order parameter. The four dimensional Hermitian Γ -matrices satisfy the anti-commuting algebra $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$, where $i, j = 1, 2, 3, 4$. The gauge field corresponding to the broken $U(1)$ symmetry is denoted by a_j , and the finite amplitude of the order parameter gives rise to the Meissner effect for a_j . The point vortex of the order parameter is introduced as

$$\Delta_1 + i\Delta_2 = |\Delta(r)| e^{in\phi}, \quad (2)$$

where n is the vorticity, and ϕ is the azimuthal angle,

and $|\Delta(r)|$ describes the radial variation of the amplitude. It has been shown via a direct calculation²⁰ that there are exactly n number of zero modes for vorticity n . Subsequently, the correspondence between the vorticity and the number of the zero energy states, has been expressed as a Z index theorem by Weinberg²². This is an index defined on the open space²³ like Callias index theorem²⁴, unlike the celebrated Atiyah-Singer index theorem²⁵, which is defined on the compact manifold. There is an additional unitary matrix $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$, which anti-commutes with H_{JR} , and ensures its spectral symmetry. All together the *five* mutually anti-commuting Γ -matrices close the Clifford algebra of the four dimensional matrices²⁶. As a consequence of the spectral symmetry, the zero energy states become eigenstates of Γ_5 , with eigenvalues ± 1 . Therefore Γ_5 defines the *chirality* of the zero modes. The Z index theorem states that

$$N_+ - N_- = n, \quad (3)$$

where N_{\pm} are respectively the number of zero energy states with chirality ± 1 . In the Altland-Zirnbauer classification scheme, this Hamiltonian belongs to the class BDI^{27,28}. The presence of the gauge field is not important for the number of zero modes or the associated index theorem. However, the existence of the gauge field is of paramount importance to realize deconfined vortices.

The Jackiw-Rossi Hamiltonian $H_{JR}[\mathbf{a}]$ can describe both insulating and superconducting states. For an insulator, a_j 's are the components of the chiral gauge field. In contrast, for a superconductor a_j 's describe the electromagnetic vector potential. For the insulating states of the charged fermions, the electromagnetic $U(1)$ symmetry is preserved and the $H_{JR}[\mathbf{a}]$ can be augmented by the electromagnetic vector potential \mathbf{A} , without altering the BDI chiral symmetry²⁹. The modified Hamiltonian is

$$H_{JR}[\mathbf{a}, \mathbf{A}] = \sum_{j=1,2} \Gamma_j(-i\partial_j + i\Gamma_3\Gamma_4 a_j + A_j) + \Delta_1\Gamma_3 + \Delta_2\Gamma_4. \quad (4)$$

The vector potential \mathbf{A} only changes the wave functions of the zero modes, without affecting its number^{29,30}. In particular for the insulators, when the vorticity is *one* ($n = 1$), there exists *single* state at zero energy. This leads to the charge fractionalization, which is relevant for the spinless fermions on the honeycomb lattice with an underlying vortex of the Kekule bond density wave order^{12,31-33}. The axial magnetic field (time-reversal preserving) can be introduced on the honeycomb lattice by deliberate wrinkling of the flake³⁴. For graphene, due to the spinful nature of the fermions, the charge fractionalization is lost. Instead, there is a competing antiferromagnetic order in the Kekule vortex core.³⁵ The fermion zero modes also occur for the vortices of the easy plane antiferromagnetic and quantum spin Hall

orders.^{29,30,36}

The fermion zero mode for the s-wave pairing of the topological insulator's surface states is also described by $H_{JR}[\mathbf{a}]$ (by ignoring the chemical potential and the Zeeman coupling). Due to the BdG nature of the quasi-particles, the single zero energy mode corresponds to a *Majorana fermion*^{37,38}. The superconductivity on the helical surface states of a topological insulator has been realized by the proximity effect³⁹, and the possibility of realizing the Majorana zero mode is also being actively pursued⁴⁰. Similar considerations can be applied for the superconducting states on the honeycomb lattice⁴¹⁻⁴⁴. However, the minimal representation of spinful Dirac fermions in graphene is 8-dimensional and to accommodate the pairings, the Nambu's doubled representation becomes 16-dimensional. In this context the associated Hamiltonian is a *direct sum* of multiple copies of the *four dimensional* Jackiw-Rossi Hamiltonian, shown in Eq. (1). The existence of multiple zero energy modes even with $n = 1$, ruins the possibility of realizing Majorana fermions⁴⁵. But, the presence of the zero modes leads to an interesting interplay of various competing order parameters inside the vortex core.³⁵

So far, we have been ignoring the effects of the finite chemical potential and the Zeeman coupling on the zero modes in a superconductor. Due to the inevitability of these couplings in a real system, it becomes important to ask which generalization of $H_{JR}[\mathbf{a}]$ still supports zero modes. The answer to this question was provided in a recent work by Herbut and Lu⁴⁶. It has been argued that one can introduce *two* additional parameters to the original Jackiw-Rossi Hamiltonian in Eq. (1), which together can support zero energy mode. The generalized Jackiw-Rossi Hamiltonian then takes the form

$$H_{gen}^{JR} = H_{JR}[\mathbf{a}] + i\Gamma_3\Gamma_4\lambda + i\Gamma_1\Gamma_2\chi. \quad (5)$$

It has been shown in Ref. 46 that there exists an anti-unitary operator (A), which anti-commutes with the total Hamiltonian H_{gen}^{JR} , guaranteeing its spectral symmetry. In the same work authors have showed that a point vortex of vorticity ± 1 can bind one zero energy state only if $\Delta_0^2 + \lambda^2 > \chi^2$, where $\Delta_0 = |\Delta(r \rightarrow \infty)|$. The physical meaning of the parameters (λ, χ) are again representation dependent. For example, if we consider spinless fermions in monolayer graphene, λ corresponds to a *chiral chemical potential*, whereas χ to the Haldane anomalous mass.⁴⁷ On the other hand, for s-wave pairing of the surface states of topological insulator, they respectively correspond to the ordinary chemical potential and the Zeeman coupling. It has been argued recently that the generalized Jackiw-Rossi Hamiltonian can support a single zero energy mode only if the vorticity is *odd*^{48,49}, and all the states are at finite energies for the even vorticity (except at $\lambda = \pm\chi$)⁵⁰. Since, the spectral symmetry with respect to Γ_5 (unitary) is now lost, the concept of chirality and Z index becomes moot. Therefore, the gen-

eralized Jackiw-Rossi Hamiltonian has a Z_2 index for the generic parameters ($\lambda \neq \pm\chi$), which can be stated as

$$\mathcal{N} = \frac{1}{2} \left(1 - (-1)^n \right) \Theta(\Delta_0^2 + \lambda^2 - \chi^2), \quad (6)$$

where \mathcal{N} and n are respectively the number of zero modes and the vorticity. A similar Z_2 index also emerges for the two dimensional $p + ip$ superconductors^{11,51}, which has been argued to be a non-relativistic limit of the H_{gen}^{JR} ^{50,52}.

In this paper we will mainly focus on the time reversal symmetric, fully gapped, momentum independent superconducting states of three dimensional Dirac fermions, which are realized in many narrow gap and gapless semiconductors. In particular, there has been a surge of interest in the superconducting states of e.g. copper intercalated bismuth selenide $Cu_xBi_2Se_3$ ⁵³, indium doped tin telluride $Sn_{1-x}In_xTe$.⁵⁴ Apart from the regular s-wave pairing, the three dimensional Dirac nature of the quasi-particle also allows the possibility of a fully gapped topologically non-trivial odd-parity pairing.⁵⁵ Even though the pairing symmetry in these materials has not been established,⁵⁶ it is known that the paired state is fully gapped and of type-II nature. This motivates us to study the possible dispersive modes along the vortex core in the mixed phase of these materials, and for concreteness we restrict ourselves in the dilute vortex limit (close to H_{c1}).

Now we provide a synopsis of our main findings in this paper.

1. For the H_{gen}^{JR} , we explicitly demonstrate the algebraic reason for the absence of the zero modes in the case of even vorticity. In addition, we reconfirm the existence of the single zero mode for odd vorticity.
2. We first consider the pairing of three dimensional massless Dirac fermions in the absence of the chemical potential and all other fermion bilinears in the particle-hole channel. When projected to the $x-y$ plane ($k_z = 0$), the Hamiltonians for both s-wave and the topological odd parity pairings map to two copies of $H_{JR}[0]$. Therefore, the Z index theorem of $H_{JR}[0]$ governs the number of Majorana zero modes.
3. When we incorporate the chemical potential, the Hamiltonian for both pairings map onto two copies of H_{gen}^{JR} . Consequently, a pair of Majorana zero modes are only found for odd vorticity. In addition, if we introduce the Dirac mass and the Zeeman couplings, the odd parity topological pairing still leads to two copies of H_{gen}^{JR} . On the other hand, the effective Hamiltonian for the s-wave pairing

does not map onto H_{gen}^{JR} in the presence of any chiral symmetry breaking perturbations in the particle-hole channel (for example scalar Dirac mass). Consequently, the zero modes are absent for s-wave pairing under generic perturbations. In contrast the zero modes for the the topological superconductors are more robust.

4. For the topological superconductor, k_z causes mixing between the two Majorana modes and converts these into a dispersing complex fermion. We provide a symmetry argument to explain why k_z does not cause any mixing between the zero modes and the finite energy Caroli-de Gennes-Matignon states. Due to this reason, the degenerate perturbation theory for k_z in the zero energy subspace provides an exact answer for the dispersion relation. We also demonstrate this through some exact solutions.
5. In the concluding sections, we consider the generalization of the zero mode problem in the context of axial superfluid. We show that the fermion zero modes can be found in the presence of the axial chemical potential and the third component of the axial vector, for odd vorticity. This leads to an interesting modification of the Callan-Harvey mechanism of anomaly cancellation.

The rest of the paper is organized as follows. In the Sec. II we demonstrate the Z_2 index associated with the generalized Jackiw-Rossi Hamiltonian. In Sec. III, we explicitly present the mapping of the three dimensional Dirac Hamiltonian with pairings to two copies of the generalized Jackiw-Rossi Hamiltonian under appropriate conditions. Sec. IV is devoted to establish the invariance of zero energy sub-space under the influence of the momentum along the vortex line. In Sec. V, we exemplify this claim by computing the dispersive modes with energies $E = \pm k_z$, as well as the vortex zero energy modes for particular choice of parameters. There we show that the dispersive modes can indeed be constructed from two dimensional zero energy modes by performing a perturbation calculation in terms of k_z . In Sec. VI, we summarize our main findings regarding the Z_2 index of H_{gen}^{JR} and its consequences for superconducting states of three dimensional Dirac fermions. In addition we propose a generalization of the Callan-Harvey model of axial superfluid, and discuss the consequences Z_2 index on anomaly equations. We relegate the details of the zero mode calculations at the special points $\lambda = \pm\chi$ to the Appendix A.

II. Z_2 INDEX OF GENERALIZED JACKIW-ROSSI HAMILTONIAN

Upon setting $\lambda = \chi = 0$ in Eq. (5), the resulting Hamiltonian conforms to the one studied originally by Jackiw and Rossi for point vortices.²⁰ It was shown in Ref. 20 and Ref. 22 that for arbitrary vorticity (n) of the point vortex, there exists precisely n number of zero energy states. This problem belongs to class BDI in the Altland-Zirnbauer classification.^{27,28} The generalized Jackiw-Rossi Hamiltonian in Eq. (5), may as well support states at zero energy depending on the vorticity and the relative strength of the parameters, λ , χ and the asymptotic value of the order parameter as $r \rightarrow \infty$ (Δ_0). Possible existence of the zero energy mode is ensured by the spectral symmetry of the Hamiltonian, H_{gen}^{JR} . Such symmetry of H_{gen}^{JR} is generated by an anti-unitary operator, namely $A = UK$, where U is the unitary operator and K is complex conjugation.^{46,57} Without any loss of generality, one can commit to a representation where Γ_1, Γ_2 are real and Γ_3, Γ_4 are imaginary.^{26,46,58} In that representation, U is the identity operator, and the anti-linear operator A is simply the complex conjugation. Furthermore, if there exists any state at precise zero energy in the spectrum of H_{gen}^{JR} , it needs to be an eigenstate of A with eigenvalue ± 1 .

A. Spectral symmetry and zero modes

Let us now focus on the zero energy modes of the Hamiltonian H_{gen}^{JR} , in Eq. (5). Since all the *four* dimensional representations of mutually anti-commuting matrices are unitarily equivalent, for our convenience we choose to work with⁵⁹

$$\begin{aligned}\Gamma_1 &= -\sigma_3 \otimes \sigma_1, \Gamma_2 = \sigma_0 \otimes \sigma_2, \Gamma_3 = \sigma_1 \otimes \sigma_1, \\ \Gamma_4 &= \sigma_2 \otimes \sigma_1, \Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = \sigma_0 \otimes \sigma_3.\end{aligned}\quad (7)$$

Let us define a four component spinor as

$$\psi^\top(\vec{x}) = (u_1, v_1, u_2, v_2)(\vec{x}), \quad (8)$$

and here we wish to solve

$$H_{gen}^{JR} \psi(\vec{x}) = 0. \quad (9)$$

In this representation, the anti-linear operator, which anti-commutes with H_{gen}^{JR} and ensures its the spectral symmetry is

$$A = i\Gamma_2 \Gamma_3 K = (\sigma_1 \otimes \sigma_3) K. \quad (10)$$

This anti-unitary operator leaves the zero energy subspace invariant. Hence, the zero energy state needs to be an eigenstate of the operator A , with eigenvalue either +1 or -1, implying

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}(\vec{x}) = \pm \begin{pmatrix} u_2^* \\ -v_2^* \\ u_1^* \\ -v_1^* \end{pmatrix}(\vec{x}). \quad (11)$$

Upon imposing the constraint on the spinor components with the + sign in last equation, the four coupled differential equations for the zero energy mode reduce to only two, which read as

$$\begin{aligned}(-i)e^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi\right)v_2^* + \Delta_r e^{-in\phi}v_2 + (\lambda + \chi)u_2^* &= 0 \\ (i)e^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi\right)u_2^* + \Delta_r e^{-in\phi}u_2 - (\lambda - \chi)v_2^* &= 0.\end{aligned}\quad (12)$$

For the ease of calculation we here chose to work with an underlying anti-vortex. Our discussion is equally applicable for point vortex. The above set of equations has been derived for $\mathbf{a} = 0$. However, the structure of these equations are not qualitatively affected by \mathbf{a} , and we will explicitly account for the gauge field in the subsequent sections. From the above equations, the imaginary factor can be removed by redefining the spinor components as

$$v_2^*(\vec{x}) = e^{i\frac{\pi}{4}}V^*(\vec{x}), \quad u_2^*(\vec{x}) = e^{-i\frac{\pi}{4}}U^*(\vec{x}). \quad (13)$$

Next we find the zero mode solution of the generalized Jackiw-Rossi Hamiltonian, shown in Eq. (9), with underlying point vortex of even and odd vorticities. One may have taken the zero energy state, as an eigenstate of A with eigenvalue -1. However, this corresponds to a phase rotation by $\exp(i\frac{\pi}{4}\Gamma_5)$, which is not an observable.⁶⁰

B. Even vorticity: $n = 2s$

Let us first consider even-vortex, i.e. $n = 2s$, where s is a positive integer. With even vorticity single-valued solution of the zero modes can only be found assuming^{41,60}

$$\begin{aligned}V(\vec{x}) &= e^{il\phi}f_1(r) + e^{im\phi}g_1(r); \\ U(\vec{x}) &= e^{ip\phi}f_2(r) + e^{iq\phi}g_2(r).\end{aligned}\quad (14)$$

The consistent solutions of Eq. (9), with the above ansatz can be obtained upon imposing the following constraints over the angular momenta

$$l + m = 2s + 1; p + q = 2s - 1; p = l - 1; q = 2s - l. \quad (15)$$

One should notice that the above constraints among different angular momenta channels of V and U arise only when λ and/or χ are nonzero. Otherwise, U and V are decoupled from each other, same as in the original work by Jackiw and Rossi.²⁰ Upon imposing these constraints over the angular momenta, we obtain the following cou-

pled differential equations for the zero energy modes

$$\left(\partial_r + \frac{l}{r}\right)f_1(r) + \Delta_r g_1(r) + (\lambda + \chi)f_2(r) = 0, \quad (16)$$

$$\left(\partial_r - \frac{l-1}{r}\right)f_2(r) + \Delta_r g_2(r) - (\lambda - \chi)f_1(r) = 0, \quad (17)$$

$$\left(\partial_r - \frac{2s-l}{r}\right)g_2(r) + \Delta_r f_2(r) - (\lambda - \chi)g_1(r) = 0, \quad (18)$$

$$\left(\partial_r + \frac{2s+1-l}{r}\right)g_1(r) + \Delta_r f_1(r) + (\lambda + \chi)g_2(r) = 0, \quad (19)$$

where

$$l = 0, 1, \dots, (2s-1). \quad (20)$$

Therefore, with even vorticity, $n = 2s$, there may exist n number of zero energy states. It is quite challenging to obtain the full analytical solution of above four equations. However, we here show that the existence of the zero energy modes can be established by studying the asymptotic behaviors of the solutions: 1) Near the origin, $r \rightarrow 0$, where the pairing order parameter Δ_r vanishes smoothly, and 2) far away from the origin, $r \rightarrow \infty$, where the Δ_r approaches a constant value Δ_0 and all the terms proportional to $1/r$ in Eq. (16)-(19) can be neglected. Next, we proceed to find the asymptotic solutions of the zero energy states.

Near origin ($r \rightarrow 0$): Neglecting the contribution from Δ_r , we arrive at the two sets of coupled differential equations. One of them reads as

$$\begin{aligned} \left(\partial_r + \frac{l}{r}\right)f_1^<(r) + (\lambda + \chi)f_2^<(r) &= 0, \\ \left(\partial_r - \frac{l-1}{r}\right)f_2^<(r) - (\lambda - \chi)f_1^<(r) &= 0. \end{aligned} \quad (21)$$

Here, we denote all the radial functions in the vicinity of the origin as $X^<(r)$, where $X = f_1, f_2, g_1, g_2$. The other set of the coupled differential equations is

$$\begin{aligned} \left(\partial_r + \frac{2s+1-l}{r}\right)g_1^<(r) + (\lambda + \chi)g_2^<(r) &= 0, \\ \left(\partial_r - \frac{2s-l}{r}\right)g_2^<(r) - (\lambda - \chi)g_1^<(r) &= 0. \end{aligned} \quad (22)$$

Solutions of the first set of equations (Eq. (21)) are

$$f_1^<(r) = \begin{cases} C_l J_{\sqrt{l}}[r\sqrt{\lambda^2 - \chi^2}] & \text{if } \lambda > \chi, \\ C_l I_{\sqrt{l}}[r\sqrt{\chi^2 - \lambda^2}] & \text{if } \lambda < \chi, \end{cases} \quad (23)$$

where C_l is an arbitrary constant. J_k and I_k are respectively the Bessel and the modified Bessel functions of first kind of order k . From the solution of $f_1^<(r)$, one can immediately find $f_2^<(r)$ from

$$f_2^<(r) = -\left(\frac{1}{\lambda + \chi}\right)\left(\partial_r + \frac{l}{r}\right)f_1^<(r). \quad (24)$$

The second set of coupled equations (Eq. (22)) yields

$$g_1^<(r) = \begin{cases} \tilde{C}_l J_{\sqrt{2s+1-l}}[r\sqrt{\lambda^2 - \chi^2}] & \text{if } \lambda > \chi, \\ \tilde{C}_l I_{\sqrt{2s+1-l}}[r\sqrt{\chi^2 - \lambda^2}] & \text{if } \lambda < \chi. \end{cases} \quad (25)$$

Here \tilde{C}_l is also an arbitrary constant. And the remaining function, $g_2^<(r)$ can be found using

$$g_2^<(r) = -\left(\frac{1}{\lambda + \chi}\right)\left(\partial_r + \frac{2s+1-l}{r}\right)g_1^<(r). \quad (26)$$

Therefore all the four radial functions in the vicinity of the origin is defined in terms of *one* (1) arbitrary constant.

Far from origin ($r \rightarrow \infty$): Next we proceed to find the radial dependence of these four functions far from the origin. Neglecting all the terms proportional to $1/r$ in Eqs. (16)-(19) we obtain a new set of four coupled differential equations

$$\begin{aligned} \partial_r f_1^>(r) + \Delta_0 g_1^>(r) + (\lambda + \chi)f_2^>(r) &= 0, \\ \partial_r g_1^>(r) + \Delta_0 f_1^>(r) + (\lambda + \chi)g_2^>(r) &= 0, \\ \partial_r f_2^>(r) + \Delta_0 g_2^>(r) - (\lambda - \chi)f_1^>(r) &= 0, \\ \partial_r g_2^>(r) + \Delta_0 f_2^>(r) - (\lambda - \chi)g_1^>(r) &= 0. \end{aligned} \quad (27)$$

Radial dependence of all the functions far away from the origin is denoted by $X^>(r)$, where $X = f_1, f_2, g_1, g_2$. Notice that far away from the origin the differential equations are independent of the angular momenta (l, m, p, q). The above four equations reduce to two sets coupled differential equations in terms of new variables, defined as⁶¹

$$F_{\pm}(r) = f_1^>(r) \pm g_1^>(r); \text{ and } G_{\pm}(r) = f_2^>(r) \pm g_2^>(r). \quad (28)$$

In terms of these variables the set of equations in Eq. (27) away from the origin becomes

$$\partial_r F_{\pm}(r) \pm \Delta_0 F_{\pm}(r) + (\lambda + \chi)G_{\pm}(r) = 0, \quad (29)$$

$$\partial_r G_{\pm}(r) \pm \Delta_0 F_{\pm}(r) - (\lambda - \chi)F_{\pm}(r) = 0. \quad (30)$$

The solution of these new equations can in general be written as

$$\begin{aligned} F_+ &= \sum_{\sigma=\pm} C_{\sigma} \exp(\alpha_{\sigma}^F r), \quad G_+ = \sum_{\sigma=\pm} C'_{\sigma} \exp(\alpha_{\sigma}^F r), \\ F_- &= \sum_{\sigma=\pm} \tilde{C}_{\sigma} \exp(\alpha_{\sigma}^G r), \quad G_- = \sum_{\sigma=\pm} \tilde{C}'_{\sigma} \exp(\alpha_{\sigma}^G r), \end{aligned} \quad (31)$$

where

$$\alpha_{\sigma}^F = -\Delta_0 + \sigma\sqrt{\chi^2 - \lambda^2}; \quad \alpha_{\sigma}^G = \Delta_0 + \sigma\sqrt{\chi^2 - \lambda^2}, \quad (32)$$

with $\sigma = \pm$. The arbitrary coefficients appearing in the solutions are related according to

$$C'_{\sigma} = -\sigma C_{\sigma} \sqrt{\frac{\chi - \lambda}{\chi + \lambda}}, \quad \tilde{C}'_{\sigma} = \sigma \tilde{C}_{\sigma} \sqrt{\frac{\chi - \lambda}{\chi + \lambda}}. \quad (33)$$

We are interested only in those solutions which decay exponentially as $r \rightarrow \infty$, so that they are normalizable. Depending on the relative strength of Δ_0 , λ and χ , the solutions can take different forms. We study each of these cases separately below.

If $0 > \chi^2 - \lambda^2 < \Delta_0^2$, $\alpha_\pm^F < 0$, but $\alpha_\pm^G > 0$. Therefore, normalizability of the solutions demands $F_- = G_- = 0$. In terms of the original functions the solutions are

$$\begin{aligned} f_1^>(r) &= g_1^>(r) = \frac{1}{2} \left(C_+ \exp(\alpha_+^F r) + C_- \exp(\alpha_-^F r) \right), \\ f_2^>(r) &= g_2^>(r) = \frac{1}{2} \sqrt{\frac{\chi - \lambda}{\chi + \lambda}} \left(C_+ \exp(\alpha_+^F r) \right. \\ &\quad \left. - C_- \exp(\alpha_-^F r) \right). \end{aligned} \quad (34)$$

If on the other hand, $\chi^2 - \lambda^2 > \Delta_0^2$, $\alpha_-^{F/G} < 0$, whereas $\alpha_+^{F/G} > 0$. Hence, for normalizable solutions $C_+ = \tilde{C}_+ = 0$, and we obtain

$$f_1^>(r) = \frac{1}{2} \left(C_- \exp(\alpha_-^F r) + \tilde{C}_- \exp(\alpha_-^G r) \right), \quad (35)$$

$$g_1^>(r) = \frac{1}{2} \left(C_- \exp(\alpha_-^F r) - \tilde{C}_- \exp(\alpha_-^G r) \right), \quad (36)$$

$$f_2^>(r) = \frac{1}{2} \sqrt{\frac{\chi - \lambda}{\chi + \lambda}} \left(C_- \exp(\alpha_-^F r) - \tilde{C}_- \exp(\alpha_-^G r) \right), \quad (37)$$

$$g_2^>(r) = \frac{1}{2} \sqrt{\frac{\chi - \lambda}{\chi + \lambda}} \left(C_- \exp(\alpha_-^F r) + \tilde{C}_- \exp(\alpha_-^G r) \right). \quad (38)$$

Therefore, irrespective of the mutual strength of λ , χ and Δ_0 , the radial part of the zero mode solutions far away from the origin is always defined by *two* (2) arbitrary constants.

Boundary conditions: To obtain self consistent solutions of the zero energy states, the spinor components need to satisfy the boundary conditions at a particular point $r = \xi$ (say). Since the solutions in the asymptotic regions are obtained by solving a second order differential equations, for each functions, we need to match values and the first derivatives of $X^<(r)$ and $X^>(r)$ at $r = \xi$, where $X = f_1, g_1, f_2, g_2$. However, for example, when we impose these boundary conditions over $f_2(r)$, it remove the arbitrariness from two out of the three constants, either (C_+, C_-, C_l) or (C_-, \tilde{C}_-, C_l) , depending on whether $\chi^2 - \lambda^2 < 0$ or $\chi^2 - \lambda^2 > \Delta_0^2$, respectively. In either situation, such elimination immediately removes the arbitrariness of the constants in the solution of $g_1^>(r)$ as well. Therefore, with two fixed constants in $g_1^>(r)$ and one arbitrary constant in the definition of $g_1^<(r)$, it is *impossible* to satisfy two of its boundary conditions.

Similarly, upon imposing the boundary conditions over $f_1(r)$, two out of either (C_+, C_-, C_l) or (C_-, \tilde{C}_-, C_l) get fixed, depending on $\chi^2 - \lambda^2 < 0$ or $\chi^2 - \lambda^2 > \Delta_0^2$, respectively. In terms of those fixed constants it is once again impossible to satisfy both the boundary conditions for $g_2(r)$. Hence, one can conclude that *for generic parameters* ($\lambda \neq \pm\chi$), *there exists no zero energy mode for H_{gen}^{JR} , when the vorticity is even.*

C. Odd vorticity: $n = 2s + 1$

Next we consider point vortex with odd vorticity, and take $n = 2s + 1$, where $s \geq 0$ is an integer. The single-valuedness and the consistency of the zero mode solutions immediately imply that out of $2s + 1$ choices for the zero modes, there is precisely one solution for which we can choose

$$V(\vec{x}) = e^{il\phi} f(r); \quad U(\vec{x}) = e^{ip\phi} g(r), \quad (39)$$

and the angular momenta l and p are related as

$$l = s + 1 = p + 1. \quad (40)$$

With these constraints over the angular momenta, the coupled differential equations for $f(r)$ and $g(r)$ read as

$$\begin{aligned} \left(\partial_r + \frac{l}{r} \right) f(r) + \Delta_r f(r) + (\lambda + \chi) g(r) &= 0, \\ \left(\partial_r + \frac{1-l}{r} \right) g(r) + \Delta_r g(r) - (\lambda - \chi) f(r) &= 0. \end{aligned} \quad (41)$$

Even though, we can exactly solve these two coupled differential equations, it is worth analyzing the existence of the zero mode solutions with the above ansatz, from their asymptotic behaviors. Later we will present the complete solution and show that these two approaches complement each other. In the vicinity of the origin ($r \rightarrow 0$), upon dropping the contribution from Δ_r , we obtain

$$f^<(r) = \begin{cases} \tilde{C}_l J_{\sqrt{l}}[r\sqrt{\lambda^2 - \chi^2}] & \text{if } \lambda > \chi, \\ \tilde{C}_l I_{\sqrt{l}}[r\sqrt{\chi^2 - \lambda^2}] & \text{if } \lambda < \chi, \end{cases} \quad (42)$$

and $g^<(r)$ can be found from the relation

$$g^<(r) = - \left(\frac{1}{\lambda + \chi} \right) \left(\partial_r + \frac{l}{r} \right) f^<(r) \quad (43)$$

At large distances ($r \rightarrow \infty$), upon setting $\Delta_r = \Delta_0$ (constant) and after dropping the terms proportional to $1/r$, we obtain

$$\begin{aligned} \partial_r f^>(r) + \Delta_0 f^>(r) + (\lambda + \chi) g^>(r) &= 0, \\ \partial_r g^>(r) + \Delta_0 g^>(r) - (\lambda - \chi) f^>(r) &= 0, \end{aligned} \quad (44)$$

which can be solved by using the ansatz $f^>(r) = c \exp(\alpha r)$, where

$$\alpha = -\Delta_0 \pm \sqrt{\chi^2 - \lambda^2} \equiv \alpha_\pm. \quad (45)$$

However, only for $\chi^2 - \lambda^2 < 0$ and consequently $\Delta_0^2 + \lambda^2 > \chi^2$, the solutions $f^>(r)$ and $g^>(r)$ are expressed in terms of *two* arbitrary constants. Then the solutions at large distances are

$$\begin{aligned} g^>(r) &= -i\sqrt{\frac{\lambda - \chi}{\lambda + \chi}} \sum_{\sigma=\pm} \sigma C_\sigma Q_\sigma(\Delta_0, \chi, \lambda, r), \\ f^>(r) &= \sum_{\sigma=\pm} C_\sigma Q_\sigma(\Delta_0, \chi, \lambda, r), \end{aligned} \quad (46)$$

where

$$Q_\sigma(\Delta_0, \chi, \lambda, r) = \exp\left(-\Delta_0 r + i\sigma\sqrt{\lambda^2 - \chi^2}\right). \quad (47)$$

Therefore, upon imposing the boundary conditions, we mentioned in the previous sub-section (say $r = \xi$), two out of three arbitrary constants for each function get fixed, while the remaining one is then set by the normalization condition. If the parameters are such that $\Delta_0^2 + \lambda^2 < \chi^2$, there is only one exponentially decaying solution, $\propto \exp(\alpha_- r)$. Therefore, each function is defined in terms of *one* arbitrary constant at large distances. Therefore, we have two arbitrary constants and two matching conditions to satisfy. After satisfying the matching conditions, there is no more arbitrary constant is left to set the overall normalization. Hence, we can have atleast one normalizable zero mode of H_{gen}^{JR} for odd vorticity and

$$\Delta_0^2 + \lambda^2 > \chi^2. \quad (48)$$

Next we show that this condition can also be obtained from the exact solution of the zero energy mode.

The exact solution of the zero energy states from Eq. (41) can be found upon assuming

$$f(r) = \tilde{f}(r)e^{-\int_0^r \Delta_{r'} dr'}, \quad g(r) = \tilde{g}(r)e^{-\int_0^r \Delta_{r'} dr'}. \quad (49)$$

The functions $\tilde{f}(r)$ and $\tilde{g}(r)$ then satisfy the following equations

$$\begin{aligned} \left(\partial_r + \frac{l}{r}\right)\tilde{f}(r) + (\lambda + \chi)\tilde{g}(r) &= 0, \\ \left(\partial_r - \frac{l-1}{r}\right)\tilde{g}(r) + (\lambda + \chi)\tilde{f}(r) &= 0, \end{aligned} \quad (50)$$

yielding

$$\tilde{f}(r) = \begin{cases} C_l J_{\sqrt{l}}[r\sqrt{\lambda^2 - \chi^2}] & \text{if } \lambda > \chi, \\ C_l I_{\sqrt{l}}[r\sqrt{\chi^2 - \lambda^2}] & \text{if } \lambda < \chi, \end{cases} \quad (51)$$

and

$$\tilde{g}(r) = -\left(\frac{1}{\lambda + \chi}\right)\left(\partial_r + \frac{l}{r}\right)\tilde{f}(r). \quad (52)$$

If we assume $\chi > \lambda$, then the solutions are expressed in terms of modified Bessel functions of first kind. However

at large distances the modified Bessel functions grow exponentially

$$I_{\sqrt{l}}(ar) \propto \frac{e^{ar}}{r\sqrt{a}} \quad (53)$$

Therefore, normalizable zero energy modes can only be found when, $\Delta_0^2 + \lambda^2 > \chi^2$, identical to the one we found in Eq. (48) by analyzing the asymptotic solutions. On the other hand, when $\lambda > \chi$, $\tilde{f}(r)$ is defined in terms of the Bessel functions of the first kind, as shown in Eq. (51). The condition in Eq. (48) is then trivially satisfied and we always find a normalizable zero mode.

Besides the above zero mode solution, there are additional $2s$ possible ansatz similar to that in Eq. (14). The angular momenta satisfy the following constraints

$$l + m = 2s + 2, p + q = 2s, p = l - 1, q = (2s + 1) - l. \quad (54)$$

Upon imposing this set of constraints over the the angular momenta, one set of coupled differential equations for the functions $f_1^<(r)$ and $f_2^<(r)$, in the vicinity of the origin assumes the identical form as in Eq. (21). Their solution can readily be found from Eq. (23), (24). The other set of equations in terms of $g_1^<(r)$ and $g_2^<(r)$ is

$$\begin{aligned} \left(\partial_r + \frac{2s + 2 - l}{r}\right)g_1^<(r) + (\lambda + \chi)g_2^<(r) &= 0, \\ \left(\partial_r + \frac{l - 2s - 1}{r}\right)g_2^<(r) - (\lambda - \chi)g_1^<(r) &= 0, \end{aligned} \quad (55)$$

yielding

$$g_1^<(r) = \begin{cases} \tilde{C}_l J_{\sqrt{2s+2-l}}[r\sqrt{\lambda^2 - \chi^2}] & \text{if } \lambda > \chi, \\ \tilde{C}_l I_{\sqrt{2s+2-l}}[r\sqrt{\chi^2 - \lambda^2}] & \text{if } \lambda < \chi. \end{cases} \quad (56)$$

One can then find $g_2^<(r)$ from the relation

$$g_2^<(r) = -\frac{1}{\lambda + \chi}\left(\partial_r + \frac{2s + 2 - l}{r}\right)g_1^<(r). \quad (57)$$

Hence, at small distances there is one arbitrary constant for each of $f_1(r), f_2(r), g_1(r)$ and $g_2(r)$. On the other hand, we have shown in the previous section that the large r behavior is independent of the angular momenta. Hence, far away from the origin the radial dependence is captured by the Eq. (34) if $\chi^2 - \lambda^2 < 0$ or Eqs. (35)-(38) if $\chi^2 - \lambda^2 > \Delta_0^2$. Therefore, following the discussion on the matching conditions at $r = \xi$, in the previous subsection, we can argue that there is no zero energy mode with multiple angular momenta ansatz.

D. Effects of gauge potential

In the above derivation, we have neglected the orbital effects of the gauge potential. It can be introduced, for

example, considering a simple profile of the magnetic field. Let us assume, that the magnetic field (applied in the z -direction, perpendicular to the plane of the vortex) is finite and constant only within a distance $r \leq \xi$, and vanishes for $r > \xi$.^{46,62} Then in the symmetric gauge, one can choose $a_\phi = \frac{r}{2\xi^2}$ when $r \leq \xi$, and $a_\phi = 1/2r$ for $r > \xi$. With such a profile of the gauge potential, the ultraviolet and the infrared asymptotic behaviors of all the functions (f_1, f_2, g_1, g_2, f, g) remain unchanged. Only significant effect will be at intermediate distance $r \sim \xi$. Hence, our derivation for the Z_2 index for the generalized Jackiw-Rossi Hamiltonian remains valid, even in the presence of the gauge field. To further demonstrate this assertion, we explicitly solve the vortex zero mode with this profile of the gauge field in Sec. V.

E. Statement of Z_2 index

After going through the above arguments, we can formally present the statement of the Z_2 index associated with the zero modes for the generalized Jackiw-Rossi Hamiltonian. For generic values of $\lambda \neq \pm\chi$, which satisfy $\Delta_0^2 + \lambda^2 > \chi^2$, there exists one normalizable zero mode when the vorticity is *odd*, while with even vorticity all the states are placed at finite energies, in accordance with Eq. (6). On the other hand, there are no normalizable zero modes for any vorticity, if $\Delta_0^2 + \lambda^2 < \chi^2$. We notice that the normalizability at infinity is governed by the uniform H_{JR}^{gen} . There is a topological phase transition or band inversion at $\Delta_0^2 + \lambda^2 = \chi^2$, and this is the reason for the absence of the normalizable zero energy state on one side.

However, there is an exception to the existence of the Z_2 index at $\lambda = \pm\chi$.⁵⁰ For these special values, the condition $\Delta_0^2 + \lambda^2 > \chi^2$ is trivially satisfied, which guarantees the normalizability of the zero modes. In the above calculation, we have assumed an anti-vortex configuration of the order parameter ($\vec{\Delta}$). With an underlying anti-vortex it can be shown that there exist precisely n number of zero energy states for arbitrary n , when $\lambda = -\chi$. Moreover, all the zero energy states are eigenstates of the the chirality operator Γ_5 , with eigenvalue $+1$. On the other hand, for $\lambda = +\chi$, once again we recover the Z_2 index for the generalized Jackiw-Rossi Hamiltonian. One can achieve the Hamiltonian

describing a point vortex defect by unitarity rotating H_{JR} in Eq. (1) by $i\Gamma_4\Gamma_5$, which changes the relative sign between Δ_1 and Δ_2 . When we perform the same operation on the generalized Jackiw-Rossi Hamiltonian in Eq. (5) it changes the sign of the $U(1)$ gauge field $\mathbf{a} \rightarrow -\mathbf{a}$, and takes $\lambda \rightarrow -\lambda$. Hence, n number of zero energy modes with underlying point vortex appears in the spectrum when $\lambda = +\chi$, whereas the Z_2 index remains unaltered for $\lambda = -\chi$. With an underlying vortex when $\lambda = +\chi$, all the n number of zero modes are eigenstate of Γ_5 , however, with eigenvalue -1 . For the detail solutions at this special values of two parameters λ and χ , readers are referred to the Appendix A.

III. THREE DIMENSIONAL MASSIVE DIRAC HAMILTONIAN WITH GAPPED PAIRINGS

Next we focus on momentum independent, time reversal symmetric, gapped paired states of the three dimensional Dirac fermions. In three spatial dimensions the Dirac quasi-particles can pair into two fully gapped superconducting states. One of them is trivial s-wave pairing, whereas the other one is parity-odd and topologically non-trivial. To study the nature of these paired states in the mixed/vortex phase, let us define an eight component Nambu-Dirac spinor as $\Psi = [\Psi_p^\top(+\vec{k}), \Psi_h^\top(-\vec{k})]$, where $\Psi_p^\top(+\vec{k}) = \Psi^\top(\vec{k})$ and $\Psi_h^\top(-\vec{k}) = \Psi_p(\vec{k})$, otherwise

$$\Psi^\top(\vec{k}) = [c_\uparrow^+(\vec{k}), c_\downarrow^+(\vec{k}), c_\uparrow^-(\vec{k}), c_\downarrow^-(\vec{k})]. \quad (58)$$

c_s^\pm corresponds to the annihilation operators for the even and the odd parity states, respectively, with the spin projections $s = \uparrow, \downarrow$. The three dimensional massive Dirac Hamiltonian in the presence of trivial s-wave (Δ_s) and odd-parity topological (Δ_T) pairings takes the form

$$H_D = \sum_{\vec{k}} \Psi^\dagger(\vec{k}) H_{gen} [\Delta_s, \Delta_T] \Psi(\vec{k}). \quad (59)$$

In order to preserve the time reversal symmetry we do not choose any relative phase between Δ_s and Δ_T . In the announced 8-dimensional Nambu-Dirac basis (Ψ), $H_{gen} [\Delta_s, \Delta_T]$ takes the form

$$\begin{aligned} H_{gen} [\Delta_s, \Delta_T] &= (\tau_0 \otimes \alpha_1)(k_x - \tau_3 \otimes I_4 a_x) + (\tau_3 \otimes \alpha_2)(k_y - \tau_3 \otimes I_4 a_y) + (\tau_0 \otimes \alpha_3)(k_z - \tau_3 \otimes I_4 a_z) + (\tau_3 \otimes I_4)\mu \\ &+ (\tau_3 \otimes \beta)m_k - (\tau_3 \otimes i\alpha_1\alpha_2)h_+ - (\tau_3 \otimes i\alpha_3\Gamma)h_- - (\tau_2 \otimes \alpha_2)\Delta_T^R + (\tau_1 \otimes \alpha_2)\Delta_T^I \\ &- (\tau_2 \otimes i\alpha_1\alpha_3)\Delta_s^R + (\tau_1 \otimes i\alpha_1\alpha_3)\Delta_s^I, \end{aligned} \quad (60)$$

where \mathbf{a} is the electromagnetic gauge potential, and the complex pairing order parameters are defined as

$$\vec{\Delta}_x = (\Delta_x^R, \Delta_x^I) = |\Delta_x| (\cos n\phi, \sin n\phi), \quad (61)$$

with $x = T, s$, and n is an integer. Then n counts the vorticity and the Hamiltonians $H_{gen} [\Delta_s, 0]$ and $H_{gen} [0, \Delta_T]$

respectively correspond to *line-vortex* defect with underlying s-wave and topologically non-trivial odd parity pairing. h_{\pm} are respectively the symmetric and the anti-symmetric combinations of the Zeeman couplings of the even (h_1) and the odd (h_2) parity bands, namely

$$h_{\pm} = \frac{1}{2}|h_1 \pm h_2|. \quad (62)$$

μ is the chemical potential and m_k is the Dirac mass. In what follows we set $m_k = m = \text{constant}$. The four dimensional Hermitian matrices are defined as

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \Gamma = \begin{pmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{pmatrix}, \quad (63)$$

which together complete the Clifford algebra of five mutually anti-commuting four dimensional matrices. Here, σ_0 is the two dimensional unit matrix and $\vec{\sigma}$ are the Pauli matrices. The other set of two dimensional matrices, $\{\tau_0, \vec{\tau}\}$, operate on the Nambu's index. Here we have ignored the anisotropy in the Fermi velocity arising from the underlying crystallographic structure, and set $v_x = v_y = v_z = v = 1$.⁶³ Next we cast the pairing Hamiltonians in the $k_z = 0$ plane with underlying point vortex defects of the topologically non-trivial odd parity pairing as orthogonal sum of two copies of the generalized Jackiw-Rossi Hamiltonian, under generic situation. Such mapping is shown to be true for the s-wave pairing, however only if there is no chiral symmetry breaking perturbations, e.g., m, h_{-} .

A. Odd-parity topological pairing

To perform the above mentioned exercise, it is worth redefining the 8-component Nambu-Dirac spinor as

$$\Psi_t^\top = [c_\uparrow^+, c_\downarrow^-, (c_\uparrow^+)^{\dagger}, (c_\downarrow^-)^{\dagger}, c_\downarrow^+, c_\uparrow^-, (c_\downarrow^+)^{\dagger}, (c_\uparrow^-)^{\dagger}](\vec{x}). \quad (64)$$

In this new basis the part of the Hamiltonian, $H_{gen}[0, \Delta_T]$ describing the point vortex in the xy -plane (i.e. with $k_z = 0$), is completely block-diagonal, whereas the k_z part is block off-diagonal. For simplicity, let us set all the orbital components of the gauge potential to zero. The total Hamiltonian with only the topological paring then takes the form

$$H_{gen}[0, \Delta_T] \rightarrow H_T^{vor} = \left(H_T^{uL} \oplus H_T^{dR} \right) + k_z \mathcal{M}_z^T, \quad (65)$$

where

$$H_T^{uL} = \gamma_5 \alpha_1 k_x + \beta \gamma_5 \alpha_2 k_y + i\beta \alpha_2 \Delta_T^R + \alpha_2 \Delta_T^I$$

$$+ \beta \alpha_3 \gamma_5 (m + h_+) + \beta (\mu + h_-), \quad (66)$$

and

$$H_T^{dR} = \gamma_5 \alpha_1 k_x - \beta \gamma_5 \alpha_2 k_y - i\beta \alpha_2 \Delta_T^R + \alpha_2 \Delta_T^I$$

$$+ \beta \alpha_3 \gamma_5 (m - h_+) + \beta (\mu - h_-). \quad (67)$$

The matrix multiplying k_z is $\mathcal{M}_z^T = \sigma_2 \otimes (i\alpha_1 \alpha_3)$. Here, we have defined as new matrix

$$\gamma_5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad (68)$$

which anti-commutes with β and Γ , but commutes with $\alpha_1, \alpha_2, \alpha_3$. Both $H_T^{uL/dR}$ in Eq. (66), and Eq. (67) assume the form of the generalized Jackiw-Rossi Hamiltonian, shown in Eq. (5). We need the following identification of the parameters $\chi \equiv m + h_+$, $\lambda \equiv \mu + h_-$ for H_T^{uL} , and $\chi \equiv m - h_+$, $\lambda \equiv \mu - h_-$ for H_T^{dR} .

Therefore, with a point vortex defect of underlying odd-parity topological paring one ends up with *two* Majorana modes, in the presence of generic perturbations. However, these Majorana modes can only be found when the vorticity of the point vortex is *odd*, as we have shown in the previous section. Since the magnetic field gets screened beyond the core of the vortex, the Zeeman couplings h_{\pm} are finite within the vortex core. Hence, two normalizable Majorana modes can be achieved only when

$$\Delta_0^2 + \mu^2 > m^2. \quad (69)$$

B. s-wave pairing

A similar exercise can also be performed with an underlying s-wave pairing, which also assumes the form of the generalized Jackiw-Rossi Hamiltonian, when $m = 0$ and $h_- = 0$. Let us now rewrite $H_{gen}[\Delta_s, 0]$ in the basis

$$\Psi_s^\top = [c_\uparrow^+, c_\downarrow^-, (c_\downarrow^+)^{\dagger}, (c_\uparrow^-)^{\dagger}, c_\downarrow^+, c_\uparrow^-, (c_\uparrow^+)^{\dagger}, (c_\downarrow^-)^{\dagger}](\vec{x}). \quad (70)$$

The eight dimensional Hamiltonian, $H_{gen}[\Delta_s, 0]$ then takes the form

$$H_{gen}[\Delta_s, 0] \rightarrow H_s^{vor} = \left(H_s^{uL} \oplus H_s^{dR} \right) + k_z \mathcal{M}_z^s, \quad (71)$$

similar to the Eq. (65). The diagonal blocks of H_s^{vor} are

$$H_s^{uL} = \gamma_5 (\alpha_1 k_x + \alpha_2 k_y) + \Delta_s^R \alpha_3 + \Delta_s^I i\alpha_3 \beta + \mu \beta$$

$$+ \gamma_5 \alpha_3 h_+ - i\beta \alpha_1 \alpha_2 m + I_4 h_-, \quad (72)$$

and

$$H_s^{dR} = \gamma_5 (\alpha_1 k_x - \alpha_2 k_y) - \Delta_s^R \alpha_3 - \Delta_s^I i\alpha_3 \beta + \mu \beta$$

$$- \gamma_5 \alpha_3 h_+ - i\beta \alpha_1 \alpha_2 m - I_4 h_-. \quad (73)$$

Here as well the k_z appears as block off-diagonal element, and the matrix multiplying k_z is $\mathcal{M}_z^s = \sigma_2 \otimes (i\alpha_1 \alpha_3)$,

identical to \mathcal{M}_z^T . Therefore, in the absence of the Dirac mass (m) and the anti-symmetric combination of the Zeeman coupling (h_-), both H_s^{uL} and H_s^{dR} are equivalent to the generalized Jackiw-Rossi Hamiltonian, shown in Eq. (5), where $\lambda = \mu$, $\chi = h_+$ for both $H_{gen}^{uL/dR}$. Hence, in the absence of any chiral symmetry breaking perturbations, the point vortex of underlying s-wave order can also support two Majorana zero modes, when the vorticity is odd. The zero energy modes in the absence of μ, h_+ , are also the eigenstates of $\beta\gamma_5 = -\sigma_2 \otimes \sigma_0$, with definite chirality. This matrix defines the chirality of massless three dimensional Dirac fermions, and should not be confused with $\Gamma_5 = \sigma_0 \otimes \sigma_3 = i\alpha_2\alpha_1$, which defines the chirality of the original Jackiw-Rossi Hamiltonian in Eq. (1). Any chiral symmetry breaking perturbation of the three dimensional Dirac fermions, such as Dirac mass (m) and h_- cause mixing among these two states and the spectrum becomes *gapped*.

IV. PERTURBATION THEORY FOR LINE VORTEX ABOUT k_z

In this section we show that the Z_2 index for the fermionic zero modes bound to the point vortex, also dictates the number of one dimensional dispersive modes along the line vortex in three spatial dimensions. The momentum along the vortex core (k_z) is shown to leave the zero energy sub-space of the underlying two dimensional Hamiltonian (multiple copies of H_{gen}^{JR}) invariant. Consequently, a first order perturbation theory for k_z is exact and the dispersive modes along the vortex core are the linear combinations of the vortex zero modes. We here prove this statement with underlying topologically non-trivial odd-parity and s-wave pairing separately.

A. Topological pairing

Let us first consider the generalized Dirac Hamiltonian with the topological pairing, $H_{gen}[0, \Delta_T]$. We have shown in the previous section that $H_{gen}[0, \Delta_T]$ is unitarily equivalent to two copies of the generalized Jackiw-Rossi Hamiltonian, when $k_z = 0$. Therefore, $H_{gen}[0, \Delta_T; k_z = 0]$ hosts Majorana zero modes for odd vorticity, under generic situation. These two zero energy states constitute a two dimensional basis, which remains invariant by any operator that commutes or anti-commutes with the Hamiltonian. If we turn off all the perturbations, namely m, μ, h_+ and h_- , then there are *four* such candidates falling into the second category. Together they close a $Cl(3) \times U(1)$ algebra. The three mutually anti-commuting matrices, closing the $Cl(3)$ sub-algebra act like standard two dimensional Pauli matrices. The remaining one, belonging to the $U(1)$ commutes with three matrices, which close the $Cl(3)$ sub-algebra. With

underlying topological pairing the $Cl(3) \times U(1)$ algebra is constituted by

$$\vec{M}_T = \{\tau_0 \otimes \alpha_3, \tau_0 \otimes \beta, \tau_0 \otimes \Gamma, \tau_0 \otimes i\alpha_1\alpha_2\}, \quad (74)$$

where the last entry belongs to the $U(1)$ part. However, due to Nambu's particle-hole doubling of the original problem, an 8-dimensional k -dependent perturbation, $a_k M_k$ can acquire a finite expectation value, only if the matrix M_k satisfies the algebraic constraint

$$M_k = \mp (\tau_1 \otimes I_4) M_k^\top (\tau_1 \otimes I_4). \quad (75)$$

In the above equation, the \mp signs depend on whether the coefficient a_k is even or odd under the parity transformation ($\mathbf{k} \rightarrow -\mathbf{k}$). If the coefficient a_k is of even parity, the only matrix satisfying the above constraint is $\tau_0 \otimes \Gamma$. On the other hand, for an odd parity a_k (e.g., linear in \mathbf{k}) the remaining three matrices can acquire finite expectation values from the zero energy sub-space. Notice that one of the matrices, $\tau_0 \otimes \alpha_3$, appearing in the $Cl(3)$ part of \vec{M}_T , multiplies k_z in H_{gen} , shown in Eq. (60). Therefore, the momentum along the vortex core k_z does not cause any mixing of the zero energy states with the rest of the spectrum. Consequently, a first order perturbation calculation in terms of k_z leads to the *exact* solution of the dispersive modes along the vortex core. The matrix $\tau_0 \otimes \alpha_3$ acts as an off-diagonal Pauli matrix in the zero energy subspace, and hence the one dimensional dispersive modes are the symmetric and the anti-symmetric combinations of the fermionic zero modes with underlying point vortex. The energies of these two dispersive modes are $E = \pm k_z$. When $k_z = 0$, we have two Majorana fermions, which hybridize via k_z and become complex fermions.

The exactness of the perturbation theory in terms of k_z is also applicable when we take into account the perturbations m, μ, h_\pm . However, not all the matrices in \vec{M}_T anti-commute with the generic Hamiltonian $H_{gen}[0, \Delta_T; k_z = 0]$. Before, we proceed to prove this statement it is worth appreciating an algebraic identity.^{29,36,64} Expectation value of an operator (\mathcal{M}) can be expressed as

$$\langle \mathcal{M} \rangle = \frac{1}{2} \left(\sum_{occupied} - \sum_{empty} \right) \Psi_E^\dagger \mathcal{M} \Psi_E, \quad (76)$$

where Ψ_E are the eigen-states of a generic Hamiltonian, \mathcal{H} . If there exists a matrix, say T , which anti-commutes with \mathcal{H} and commutes with \mathcal{M} , the above mentioned sum is restricted to the zero energy subspace. When $m = \mu = h_\pm = 0$, and $\mathcal{M} = (\tau_0 \otimes \alpha_3)k_z$, one can choose $T = \tau_0 \otimes i\alpha_1\alpha_2$. When m, μ, h_\pm are finite, we can still find a matrix, T with requisite criteria, for the chosen $\mathcal{M} = (\tau_0 \otimes \alpha_3)k_z$. If all the perturbations m, μ, h_\pm are nonzero, we cannot find any unitary matrix for T . Rather there is an anti-unitary operator (A_T), namely $(\tau_1 \otimes i\beta\alpha_1\alpha_2) K$, where K is the complex

conjugation, which anti-commutes with the Hamiltonian $H_{gen}[0, \Delta_T; k_z = 0]$ and commutes with $(\tau_0 \otimes \alpha_3)k_z$. Therefore, we can choose $T = A_T$. Hence, the one dimensional dispersive modes are always the symmetric and the anti-symmetric combinations of the zero energy modes bound to the point vortex. Even in the presence of the gauge fields, we can still choose $T = A_T$, and the above conclusions remain unaltered.

B. s-wave pairing

A similar conclusion can be arrived at even with an underlying s-wave pairing if we turn off all the chiral symmetry breaking perturbations, for example m, h_- . Furthermore, when μ, h_+ is set to zero the $Cl(3) \otimes U(1)$ algebra of the matrices, anti-commuting with the Hamiltonian $H_{gen}[\Delta_s, 0; k_z = 0]$ is constituted by

$$\vec{M}_s = \{\tau_0 \otimes \Gamma, \tau_0 \otimes \alpha_3, \tau_3 \otimes \beta, \tau_0 \otimes i\alpha_1\alpha_2\}, \quad (77)$$

where the last entry belongs to the $U(1)$ part. Appearance of the matrix $\tau_0 \otimes \alpha_3$ in the $Cl(3)$ part of \vec{M}_s , immediately guarantees that the dispersive one dimensional modes can be obtained by performing a perturbative calculation over k_z within the two dimensional basis spanned by the fermionic zero energy modes due to a point vortex in the $k_z = 0$ plane. It can also be confirmed from the Eq. (76), upon choosing $T = \tau_0 \otimes i\alpha_1\alpha_2$. Otherwise, the matrix $\tau_0 \otimes \alpha_3$ acts as the *diagonal* Pauli matrix. Hence, the dispersive modes, with energies $E = \pm k_z$, are identical to the fermionic zero mode due to the point vortex. Let us now incorporate a finite chemical potential (μ) and the symmetric Zeeman coupling h_+ . One can then choose $T = (\tau_2 \otimes i\beta\alpha_1\alpha_2) K$, for $\mathcal{M} = (\tau_0 \otimes \alpha_3)k_z$. In conjunction with such choice of T , Eq. (76) guarantees that the dispersive modes with $E = \pm k_z$, are exactly the two fermionic zero modes bound to the point vortex.

It is worth mentioning that one of matrices in the $Cl(3)$ subgroup, namely $\tau_3 \otimes \beta$ appears in H_{gen} in Eq. (60) with the Dirac mass (m). This matrix also satisfies the condition in Eq. (75), if its coefficient is momentum independent. Therefore, the internal structure of the zero energy sub-space shows that the Dirac mass is sufficient to cause splitting of these two states and place them at finite energies, $\pm m$.

V. EXACT AND PERTURBATIVE SOLUTIONS OF LINE VORTEX

The internal structure of the zero energy modes of the point vortex allowed us to show that the one dimensional dispersive modes along the core of the vortex can be

constructed from the Majorana zero modes due to the point vortex. Here, we first present the exact solutions of the dispersive modes for line-vortex as well as the zero mode solutions of the point vortex, for particular choices of the parameters in $H_{gen}[\Delta_s, \Delta_T]$. Then we show that the one dimensional dispersive modes with energies $E = \pm k_z$, are either linear combinations of (for the topological pairing) or exactly (for the s-wave pairing) the zero modes for the point vortex. Some particular limits of this problem has been considered previously in Refs. 16–19.

A. Topological pairing with $\mu = h_+ = 0$

Let us first present the solution of the dispersive one dimensional modes along the vortex core with underlying topological pairing, when $\mu = h_+ = 0$. The coupled differential equations for gapless modes along the vortex core read as

$$(m + h_-)\Lambda_\uparrow^+ + (-i)e^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi + a_\phi\right)\Lambda_\downarrow^- + k_z\Lambda_\uparrow^- + \Delta_r e^{-i\phi}\left(\Lambda_\downarrow^-\right)^\dagger = E\Lambda_\uparrow^+, \quad (78)$$

$$-(m - h_-)\Lambda_\downarrow^- + (-i)e^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi - a_\phi\right)\Lambda_\uparrow^+ - k_z\Lambda_\downarrow^+ - \Delta_r e^{-i\phi}\left(\Lambda_\uparrow^+\right)^\dagger = E\Lambda_\downarrow^-, \quad (79)$$

$$(m - h_-)\Lambda_\downarrow^+ + (-i)e^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi - a_\phi\right)\Lambda_\uparrow^- - k_z\Lambda_\downarrow^- - \Delta_r e^{-i\phi}\left(\Lambda_\uparrow^-\right)^\dagger = E\Lambda_\downarrow^+, \quad (80)$$

$$-(m + h_-)\Lambda_\uparrow^- + (-i)e^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi + a_\phi\right)\Lambda_\downarrow^+ + k_z\Lambda_\uparrow^+ + \Delta_r e^{-i\phi}\left(\Lambda_\downarrow^+\right)^\dagger = E\Lambda_\uparrow^-. \quad (81)$$

The remaining four equations can readily be found by taking the Hermitian conjugate of the above four. Δ_r captures the radial variation of the superconducting gap, which in the presence of line-vortex vanishes smoothly as $r \rightarrow 0$, and saturates at Δ_0 (constant) as $r \rightarrow \infty$, otherwise arbitrary. Next we wish to find the solution for two different cases, i) $E = k_z$ and ii) $E = -k_z$. In the former situation, the condition $E = k_z$ leads to a set of constraints among the various components of the Nambu-Dirac spinor, namely

$$\Lambda_\uparrow^- = \Lambda_\uparrow^+, \Lambda_\downarrow^- = -\Lambda_\downarrow^+, (\Lambda_\uparrow^-)^\dagger = (\Lambda_\uparrow^+)^\dagger, (\Lambda_\downarrow^-)^\dagger = -(\Lambda_\downarrow^+)^\dagger. \quad (82)$$

Upon imposing these constraints the above four equations reduce to only two, and they read as

$$(-i)e^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi + a_\phi\right)\Lambda_\downarrow^- + (m + h_-)\Lambda_\uparrow^+ + \Delta_r e^{-i\phi}\left(\Lambda_\downarrow^-\right)^\dagger = 0, \quad (83)$$

$$(-i)e^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi - a_\phi\right)\Lambda_\uparrow^+ - (m - h_-)\Lambda_\downarrow^- - \Delta_r e^{-i\phi}\left(\Lambda_\uparrow^+\right)^\dagger = 0. \quad (84)$$

These two coupled equations can be solved using the ansatz

$$\lambda_\uparrow^+ = \mathcal{R}(r) e^{-i\phi} g(r); \lambda_\downarrow^- = \mathcal{R}^*(r) f(r), \quad (85)$$

where $\mathcal{R}(r) = \exp\left(-i\frac{\pi}{4} - \int_0^r \Delta_{r'} dr'\right)$. We here take the profile of the vector potential \mathbf{a} to be same as in Sec. II D. With such a profile of \mathbf{a} , one obtains

$$\begin{aligned} g(r) &= c_1 I_1 \left[r \sqrt{m^2 - h^2} \right], \\ f(r) &= c_1 \sqrt{\frac{m+h}{m-h}} I_0 \left[r \sqrt{m^2 - h^2} \right], \end{aligned} \quad (86)$$

when $r \ll \xi$. Outside the core of the vortex ($r > \xi$)

$$\begin{aligned} g(r) &= c_3 I_{1/2} \left[r|m| \right] + c_4 I_{-1/2} \left[r|m| \right], \\ f(r) &= c_4 I_{1/2} \left[r|m| \right] + c_3 I_{-1/2} \left[r|m| \right]. \end{aligned} \quad (87)$$

The solutions and their first derivatives need to be matched at $r = \xi$, where the solutions for $r < \xi$ can be found by replacing $h^2 \rightarrow h^2 + 1/2\lambda^2$.⁴⁶ It eliminates two out of three arbitrary constants from $f(r)$ and $g(r)$, while the remaining one is fixed by the normalization condition. The complete solution in the basis of 8-component Nambu-Dirac spinor shown in Eq. (58) reads as

$$|E = +k_z\rangle = e^{-i\frac{\pi}{4} + ik_z z} e^{-\int_0^r \Delta_{r'} dr'} \begin{pmatrix} g(r)e^{-i\phi} \\ if(r) \\ -g(r)e^{-i\phi} \\ if(r) \\ ig(r)e^{i\phi} \\ f(r) \\ -ig(r)e^{i\phi} \\ f(r) \end{pmatrix}. \quad (88)$$

On the other hand, when we wish to solve the dispersive mode with $E = -k_z$, the components of Nambu spinor are related according to

$$\Lambda_\uparrow^- = -\Lambda_\uparrow^+, \Lambda_\downarrow^- = \Lambda_\downarrow^+, (\Lambda_\uparrow^-)^\dagger = -(\Lambda_\uparrow^+)^\dagger, (\Lambda_\downarrow^-)^\dagger = (\Lambda_\downarrow^+)^\dagger. \quad (89)$$

These constraints also reduce the number of coupled equations from *four* to *two*. Otherwise, the solution in terms of Λ_\uparrow^+ and Λ_\downarrow^- are identical to that for $E = +k_z$ mode. In terms of 8-component Nambu-Dirac fermions the dispersive gapless mode with $E = -k_z$ assumes the form

$$|E = -k_z\rangle = e^{-i\frac{\pi}{4} + ik_z z} e^{-\int_0^r \Delta_{r'} dr'} \begin{pmatrix} g(r)e^{-i\phi} \\ -if(r) \\ g(r)e^{-i\phi} \\ if(r) \\ ig(r)e^{i\phi} \\ -f(r) \\ ig(r)e^{i\phi} \\ f(r) \end{pmatrix}, \quad (90)$$

where the radial functions $f(r)$ and $g(r)$ are identical to the ones in Eq. (86) and Eq. (87), for $r \ll \xi$ and $r > \xi$, respectively.

Next we present the solution of the fermionic zero modes with an underlying point vortex, when $\mu = h_+ = 0$. With this particular choice of parameters, one set of coupled differential equations for the zero mode reads as

$$\begin{aligned} e^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi + a_\phi\right)\Lambda_\downarrow^- + i(m + h_-)\Lambda_\uparrow^+ \\ + i\Delta_r e^{-i\phi}\left(\Lambda_\downarrow^-\right)^\dagger = 0, \end{aligned} \quad (91)$$

$$\begin{aligned} ie^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi - a_\phi\right)\Lambda_\uparrow^+ + (m - h_-)\Lambda_\downarrow^- \\ + \Delta_r e^{-i\phi}\left(\Lambda_\uparrow^+\right)^\dagger = 0. \end{aligned} \quad (92)$$

The fermionic zero obtained by solving this set of equations, in the basis of 8-component Nambu-Dirac spinor reads as

$$|\Psi_1^0\rangle = e^{-i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'} \begin{pmatrix} g(r)e^{-i\phi} \\ 0 \\ 0 \\ if(r) \\ ig(r)e^{i\phi} \\ 0 \\ 0 \\ f(r) \end{pmatrix}, \quad (93)$$

where the two radial functions $f(r)$ and $g(r)$ takes the identical form as in Eq. (86) and Eq. (87) for $r \ll \xi$ and $r > \xi$, respectively. The other set of coupled differential equations for the zero mode reads as

$$\begin{aligned} e^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi + a_\phi\right)\Lambda_\downarrow^+ - i(m + h_-)\Lambda_\uparrow^- \\ + i\Delta_r e^{-i\phi}\left(\Lambda_\downarrow^+\right)^\dagger = 0, \end{aligned} \quad (94)$$

$$(i)e^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi - a_\phi\right)\Lambda_\uparrow^- - (m - h_-)\Lambda_\downarrow^+ + \Delta_r e^{-i\phi}\left(\Lambda_\uparrow^-\right)^\dagger = 0. \quad (95)$$

The corresponding zero energy state takes the form

$$|\Psi_2^0\rangle = e^{-i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'} \begin{pmatrix} 0 \\ -if(r) \\ g(r)e^{-i\phi} \\ 0 \\ 0 \\ -f(r) \\ ig(r)e^{i\phi} \\ 0 \end{pmatrix}, \quad (96)$$

where $f(r)$ and $g(r)$ are identical as for the other zero mode, $|\Psi_1^0\rangle$.

It is now evident that the solution for two one dimensional dispersive modes, $|E = +k_z\rangle$ and $|E = -k_z\rangle$ in Eqs. (88) and (90), are respectively the symmetric and the anti-symmetric combination of two fermion zero mode in Eqs. (93) and (96), in the presence of point vortex.

B. s-wave paring with $m = \mu = h_\pm = 0$

Let us now consider a line vortex along the z -direction with underlying s-wave paring. The dispersive mode can be solved analytically when we set $m = \mu = h_\pm = 0$. Furthermore, we also turn off the orbital contribution of the gauge field, i.e. $a_\phi = 0$. Then the coupled differential equations read as

$$\begin{aligned} (-i)\left(\partial_r - \frac{i}{r}\partial_\phi\right)\Lambda_\downarrow^- + k_z\Lambda_\uparrow^- + e^{-i\phi}\Delta_r\left(\Lambda_\downarrow^+\right)^\dagger &= E\Lambda_\uparrow^+, \\ (-i)\left(\partial_r + \frac{i}{r}\partial_\phi\right)\Lambda_\uparrow^- - k_z\Lambda_\downarrow^- + e^{-i\phi}\Delta_r\left(\Lambda_\uparrow^+\right)^\dagger &= E\Lambda_\downarrow^+, \\ (-i)\left(\partial_r - \frac{i}{r}\partial_\phi\right)\Lambda_\downarrow^+ + k_z\Lambda_\uparrow^+ + e^{-i\phi}\Delta_r\left(\Lambda_\downarrow^-\right)^\dagger &= E\Lambda_\uparrow^-, \\ (-i)\left(\partial_r + \frac{i}{r}\partial_\phi\right)\Lambda_\uparrow^+ - k_z\Lambda_\downarrow^+ + e^{-i\phi}\Delta_r\left(\Lambda_\uparrow^-\right)^\dagger &= E\Lambda_\downarrow^-. \end{aligned} \quad (97)$$

The remaining four equations are the Hermitian conjugate of above four. Upon imposing the constraints over the spinor components, as shown in Eq. (82), the above four equation reduces to *two* decoupled equations. For $E = +k_z$ they read as

$$ie^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi\right)\Lambda_\downarrow^- + e^{-i\phi}\Delta_r\left(\Lambda_\downarrow^-\right)^\dagger = 0, \quad (98)$$

$$ie^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi\right)\Lambda_\uparrow^+ + e^{-i\phi}\Delta_r\left(\Lambda_\uparrow^+\right)^\dagger = 0. \quad (99)$$

The first equation yields

$$\Lambda_\downarrow^- = C_-^+ e^{-i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'}, \quad (100)$$

whereas the second one gives

$$\Lambda_\uparrow^+ = \frac{C_+^+}{r} e^{-i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'}, \quad (101)$$

where C_\pm^+ are the arbitrary constants. However, to keep the second solution well behaved near the origin, we have to set $C_+^+ = 0$, therefore $\Lambda_\uparrow^+ \equiv 0$. On the other hand, the constraints in Eq. (89) yield the following two decoupled equations for $E = -k_z$

$$(-i)e^{-i\phi}\left(\partial_r - \frac{i}{r}\partial_\phi\right)\Lambda_\downarrow^- + e^{-i\phi}\Delta_r\left(\Lambda_\downarrow^-\right)^\dagger = 0, \quad (102)$$

$$(-i)e^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi\right)\Lambda_\uparrow^+ + e^{-i\phi}\Delta_r\left(\Lambda_\uparrow^+\right)^\dagger = 0, \quad (103)$$

giving

$$\Lambda_\downarrow^- = (i) C_-^- e^{-i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'}, \quad (104)$$

and

$$\Lambda_\uparrow^+ = \frac{C_+^-}{r} e^{i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'}, \quad (105)$$

where C_\pm^- are also arbitrary constants. For the solutions to be well behaved in the vicinity of the origin we need to set $C_+^- = 0$. In conjunction with the constraints, the above solutions in Eq. (100) and Eq. (104), yield two dispersive modes with underlying s-wave order

$$|+k_z\rangle = C_-^+ \mathcal{R}(r, z) \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -i \\ 0 \\ i \end{pmatrix}, | -k_z\rangle = C_-^- \mathcal{R}(r, z) \begin{pmatrix} 0 \\ i \\ 0 \\ i \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad (106)$$

where $\mathcal{R}(r, z) = \exp(-i\frac{\pi}{4} - \int_0^r \Delta_{r'} dr' + ik_z z)$.

Next we proceed to obtain the zero energy ($E = 0$) modes with an underlying point vortex of s-wave paring in the $k_z = 0$ plane. It can readily be solved from Eq. (97) upon setting $E = 0, k_z = 0$. In light of the above discussion, readers can convince themselves that $\Lambda_\uparrow^+, \Lambda_\uparrow^-$ and their conjugates in the zero energy subspace are proportional to $1/r$. Therefore, these components are exactly zero for the normalizable zero mode solutions. The remaining components are finite for two zero energy modes. For one Majorana mode

$$\Lambda_\downarrow^- = \Lambda_\downarrow^+ = C_-^+ e^{i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'}, \quad (107)$$

and, for the other one

$$\Lambda_{\downarrow}^{-} = -\Lambda_{\downarrow}^{+} = C_{-}^{-} e^{-i\frac{\pi}{4}} e^{-\int_0^r \Delta_{r'} dr'}. \quad (108)$$

k_z acts like $\sigma_3 = \text{diag.}(1, -1)$ matrix in the zero energy sub-space. Therefore, it does not cause any mixing between two Majorana modes. Consequently, the gapless modes are same as the vortex zero modes, multiplied by the plane wave factor $\exp ik_z z$.

VI. DISCUSSIONS

In this paper we have demonstrated that the original Jackiw-Rossi Hamiltonian describing a point vortex defect in two spatial dimensions (see Eq. (1)), can be augmented by two additional terms (see Eq. (5)), which still possesses a spectral symmetry. Consequently, the generalized Jackiw-Rossi Hamiltonian of Eq. (5) may support zero modes. For generic values of the perturbation parameters satisfying the condition in Eq. (48), we obtain a single zero mode only for the odd vorticity. In contrast, there are no zero modes for the even vorticity, for generic perturbations. To demonstrate the emergence of the Z_2 index for the zero modes, we have employed a method of matching asymptotic solutions, which correctly captures all the known results within a single framework. We have also found that there exist special values $\lambda = \pm\chi$, for which there are n number of zero modes respectively for a vortex and an antivortex of vorticity n .

One of the main goal of this paper is to determine the number of gapless one dimensional modes along the line vortex of a gapped paired states of three dimensional Dirac quasiparticles. In order to answer this question, we have mapped the problem into the determination of the number of vortex zero modes of appropriate H_{gen}^{JR} 's. Through this procedure, we have succeeded in showing that the number of gapless modes is also dictated by the Z_2 index of H_{gen}^{JR} 's. We have exemplified this by considering the topological odd parity and trivial s-wave pairings. If the underlying Dirac fermions are massless, then in the absence of the chemical potential and the Zeeman couplings, both types of pairing lead to two copies of appropriate H_{JR} for $k_z = 0$. Consequently, the number of the gapless modes is governed by the Z index theorem of Weinberg. When generic perturbations are considered, only the topological pairing sustains *two* gapless modes for odd vorticity, in accordance with the Z_2 index.

Our calculations for isolated vortex can be extended in perturbation theory set up for multiple vortices in the dilute vortex limit. In the presence of multiple vortices, if we again consider $k_z = 0$, there will be tunneling within each copy of H_{gen}^{JR} 's (see Eq. (66) and

Eq. (67)) for topological superconductor. As far as the topology of the order parameter field is concerned, there is no difference between a n-vortex and widely separated n-number of single vortices. Our analysis now suggests an interesting *even-odd* effect based on the Z_2 index. When the net vorticity is odd, the gapless state will survive the tunneling effects. In contrast, the gapless state will be absent for a net even vorticity. This consideration can also be extended to non-dilute limit following the strategy in Ref. 65.

Our analysis can be applied to an interesting problem of chiral anomaly for the gapless one dimensional modes along the vortex line of an axial superfluid, which was considered by Callan & Harvey.¹³ They have considered the following model

$$H_{ax} = \sum_{j=1}^3 \gamma_0 \gamma_j (-i\partial_j - eA_j) + \Delta(r) \gamma_0 \exp(i\theta \gamma_0 \gamma_5), \quad (109)$$

where we have used five mutually anti-commuting γ -matrices, and A_j is the electromagnetic vector potential, and e is the electron's charge. When, we consider a line n-vortex of the axial superfluid order parameter along the z direction ($\theta = n\phi$), there are n gapless chiral one dimensional modes. This number of modes is tied with the Weinberg's Z index theorem for the underlying Jackiw-Rossi problem in the $x - y$ plane. The chiral one dimensional modes in the presence of the electric field along the z direction, give rise to a non-dissipative electric current along the z direction, determined by one dimensional chiral anomaly $j_z = n \times e^2 E_z / (2\pi)$. This current in turn is supplied radially from the bulk into the vortex core, which is captured by the following axion electrodynamics term

$$\mathcal{L}_{axion} = -\frac{e^2}{8\pi^2} \int d^4x \epsilon^{\mu\nu\rho\lambda} \partial_\mu \theta A_\nu \partial_\rho A_\lambda. \quad (110)$$

Now we may add various fermion bilinears to the above model, which can still support gapless modes along the vortex line. According to the construction of H_{gen}^{JR} in the previous sections, we can add $\gamma_5 \lambda$ and $i\gamma_1 \gamma_2 \chi$, which respectively describe an axial chemical potential, and the third component of the space-like axial vector (these terms break the Lorentz and CPT symmetries). Now there is a single gapless mode only for the odd vorticity, if $\Delta_0^2 + \lambda^2 > \chi^2$. The value of the amplitude Δ at radial infinity has been chosen to be Δ_0 . Consequently, we can find non-dissipative current only for the odd vorticity, under generic values of these parameters. Accordingly the bulk axion term, which is usually computed through the Goldstone-Wilczek formula,² has to be modified to capture the Z_2 index.

The condition $\Delta_0^2 + \lambda^2 > \chi^2$ has a simple physical meaning in terms of the uniform model's band structure. If $\chi = 0$, the Kramer's degeneracy is lifted by λ , but

keeping the spectrum fully gapped. On the other hand, for $\lambda = 0$, the spectrum is fully gapped only when $\Delta > \chi$. For $\chi > \Delta$ we have a Weyl semi-metal phase, which does not support the vortex zero modes. In the Weyl semi-metal phase, there are chiral surface states, which lead to anomalous Hall conductivity and chiral magnetic conductivity.⁹ The anomalous transport properties in the gapless phase are also captured by appropriate axion electrodynamics terms, which are not related to vortex zero modes. This is also interesting to note that the number of gapless modes also controls an associated gravitational anomaly formula, and our work suggests its Z_2 modification in the presence of λ and/or χ . We also note that the axial vector χ breaks the spatial rotational symmetry, and for this reason the zero modes can only be found for a line vortex aligned with the axial vector.

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Appendix A: Zero energy states at $\lambda = \pm\chi$

We here present the detail solutions of the zero energy states with an underlying anti-vortex for $\lambda = \pm\chi$. When $\lambda = -\chi$, one can write the the equation for the zero energy modes for even-vorticity ($n = 2s$) from Eq. (16)-(19). Written slightly differently, they read as

$$\begin{aligned} \left(\partial_r + \frac{p+1}{r}\right)f_1(r) + \Delta_r g_1(r) &= 0, \\ \left(\partial_r - \frac{p}{r}\right)f_2(r) + \Delta_r g_2(r) - 2\lambda f_1(r) &= 0, \\ \left(\partial_r - \frac{2s-1-p}{r}\right)g_2(r) + \Delta_r f_2(r) - 2\lambda g_1(r) &= 0, \\ \left(\partial_r + \frac{2s-p}{r}\right)g_1(r) + \Delta_r f_1(r) &= 0, \end{aligned} \quad (\text{A1})$$

where p is a positive definite integers, and takes the values

$$p = 0, 1, \dots, 2s-1. \quad (\text{A2})$$

In the vicinity of the origin, where $\Delta_r \rightarrow 0$, $f_1^<(r) = \tilde{c}_1 r^{-(p+1)}$, and $g_1^<(r) = \tilde{c}_2 r^{-(2s-p)}$. Hence, normalizable

zero energy modes can only be found when $\tilde{c}_1 = \tilde{c}_2 = 0$. On the other hand, in the vicinity of the origin

$$f_2^<(r) = c_1 r^p, \quad g_2^<(r) = c_2 r^{2s-1-p}, \quad (\text{A3})$$

are well behaved functions for all p given in Eq. (A2). Far away from the origin, these two functions are $f_2^>(r) = g_2^>(r) = c \exp(-\Delta_0 r)$. Two out three arbitrary constants (c_1, c_2, c), can be fixed by imposing the boundary conditions

$$f_1^<(r = \xi) = f_1^>(r = \xi), \quad g_1^<(r = \xi) = g_1^>(r = \xi), \quad (\text{A4})$$

while the remaining one is determined by the overall normalization factor. Hence, there are $n = 2s$ number of zero energy states with underlying anti-vortex, when $\lambda = -\chi$. In terms of the original functions in Eq. (14), $V(\vec{x}) = 0$ for the zero energy modes. Therefore, the zero energy modes are eigenstate of the operator $\Gamma_5 = \sigma_0 \otimes \sigma_3$ with eigenvalue $+1$. These solutions match exactly with the ones found by Jackiw and Rossi, in the absence of gauge fields.²⁰

If, on the other hand, $\lambda = \chi$, we have

$$\begin{aligned} f_2^<(r) &= \tilde{c}_2 r^p, \quad f_1^<(r) = -\frac{\tilde{c}_2 \lambda}{p+1} r^{p+1}, \\ g_2^<(r) &= c'_2 r^{2s-1-p}, \quad g_1^<(r) = \frac{c'_2 \lambda}{p-2s} r^{2s-p}, \end{aligned} \quad (\text{A5})$$

where \tilde{c}_2, c'_2 are arbitrary constants. Far aways from the origin the functions behave as

$$\begin{aligned} f_2^>(r) &= g_2^>(r) = c \exp(-\Delta_0 r), \\ f_1^>(r) &= g_1^>(r) = -2c\lambda r \exp(-\Delta_0 r), \end{aligned} \quad (\text{A6})$$

where c is also an arbitrary constant. After satisfying the boundary conditions, for example the one shown in Eq. (A4), two out of three arbitrary constants, (\tilde{c}_2, c'_2, c) are fixed. With only one arbitrary constant, it is now impossible to satisfy two similar boundary conditions for $f_2(r)$ and $g_2(r)$. Therefore, when $\lambda = \chi$, there is no zero energy state when the underlying anti-vortex has even vorticity.

Let us now consider the anti-vortex with odd vorticity, $n = 2s + 1$. We first focus on $2s$ ansatz of the form Eq. (14). The equation of the zero energy modes then reads as

$$\begin{aligned} \left(\partial_r + \frac{p+1}{r}\right)f_1(r) + \Delta_r g_1(r) &= 0, \\ \left(\partial_r - \frac{p}{r}\right)f_2(r) + \Delta_r g_2(r) - 2\lambda f_1(r) &= 0, \\ \left(\partial_r - \frac{2s-p}{r}\right)g_2(r) + \Delta_r f_2(r) - 2\lambda g_1(r) &= 0, \\ \left(\partial_r + \frac{2s-p+1}{r}\right)g_1(r) + \Delta_r f_1(r) &= 0, \end{aligned} \quad (\text{A7})$$

when $\lambda = -\chi$. For the normalizable zero energy modes, we find $f_1(r) = g_1(r) = 0$. The remaining two functions in the vicinity of the origin read as

$$f_2^<(r) = c_1 r^p, \quad g_2^<(r) = c_2 r^{2s-p}, \quad (\text{A8})$$

and far away from origin they are

$$f_2^>(r) = g_2^>(r) = c \exp(-\Delta_0 r). \quad (\text{A9})$$

Hence, for $\lambda = -\chi$, there exists $2s$ number of zero energy states with an underlying anti-vortex of vorticity $2s + 1$ of the form Eq. (14). These solutions are also identical to the ones one find when $\lambda = \chi = 0$.²⁰ However, when $\lambda = \chi$, there is no zero energy mode of the form Eq. (14).

Next we solve the zero energy mode with the ansatz of the form Eq. (39), when $\lambda = \pm\chi$. One can also write the Eq. (41) for the zero energy states as

$$\begin{aligned} \left(\partial_r + \frac{p+1}{r} \right) f(r) + \Delta_r f(r) + (\lambda + \chi)g(r) &= 0, \\ \left(\partial_r - \frac{p}{r} \right) f(r) + \Delta_r f(r) - (\lambda - \chi)g(r) &= 0, \end{aligned} \quad (\text{A10})$$

where $p = (n - 1)/2$, and $n(\text{odd})$ is the vorticity. For $\lambda = -\chi$, the radial functions of the zero modes are given by

$$f(r) = 0, \quad g(r) = c r^p \exp\left(-\int_0^r \Delta_{r'} dr'\right), \quad (\text{A11})$$

where c is an arbitrary constant, similar to the one found in original work by Jackiw-Rossi.²⁰ With this particular ansatz, we also find normalizable zero energy modes even when $\lambda = \chi$. The radial functions then read as

$$\begin{aligned} f(r) &= -c \left(\frac{\lambda}{p+1} \right) r^{p+1} \exp\left(-\int_0^r \Delta_{r'} dr'\right), \\ g(r) &= c \exp\left(-\int_0^r \Delta_{r'} dr'\right). \end{aligned} \quad (\text{A12})$$

Therefore, the zero energy mode of the form Eq. (39), exists whether $\lambda = \chi$ or $\lambda = -\chi$. For $n = 1$ or $p = 0$, this solution matches exactly with the one shown in Ref. 46.

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